A numerical study of resonant regimes near the points of codimension-2 bifurcation in the Couette –Taylor problem

Ovchinnikova S. N., Morshneva I. V.

South Federal University
Dept. of Math., Mech. and Comp. Sciences
Milchakova st., 8a, 344090, Rostov-on-Don, Russia
morsh@math.rsu.ru

Abstract :

The investigation of codimension-2 bifurcations in the systems with cylindric symmetry enables us to find out new types of secondary regimes branching off from the symmetric regimes. The central manifold approach combined with the reduction to the normal form lead to the so called amplitude systems.These ODE systems describe the nonlinear interaction between the neutral modes and include always several nonlinear terms due to so called intrinsic resonances. However, sometimes additional resonances appear. We present the results of investigation of interaction between two nonaxisymmetric neutral modes in the Couette–Taylor problem in the intrinsic resonances case (Res 0) and one of six additional resonances (Res 3).

Key-words :

resonances; bifurcation; amplitude equations

1 Introduction

We consider bifurcations in the Couette flow between rigid co-axial rotating cylinders. The system in consideration is invariant with respect to the linear orthogonal action of the Lie group $\mathcal{G} = SO(2) \times O(2)$. For all that the Couette flow is the most symmetric — it is invariant with respect to all transformations of the group $\mathcal{G}$. The principal part of the present work is just concerned with this generic case, while the specific character of the Couette flow becomes essential only when defining numerical values of coefficients of the amplitude systems.

In the space of parameters of the Couette–Taylor problem there are two-parametric families (surfaces) and one-parametric families (curves) filled by points of intersections of bifurcations, which correspond to various nonaxisymmetric neutral modes. Near the point of intersection a strong interaction of all respective almost-neutral modes becomes possible. A passage to the central manifold makes it possible to describe behavior of the system near an equilibrium, with near-critical values of parameters, by means of a nonlinear system of amplitude equations (Yudovich (1986) – Chossat *et al.* (1994)). Reducing the system on central manifold to a normal form is achieved simply by averaging over the "fast time". For all that there always remain in the equations some terms corresponding to "intrinsic" resonances in the system. However, additional resonances are possible too, that leads appearance of some new nonlinear interactions and new terms in the amplitude equations.

In Yudovich (1986) – Chossat *et al.* (1994) a system of amplitude equations for the Taylor–Couette problem is constructed, and various possible flow regimes for the parameters close to the point of intersections are studied. In these papers only the basic resonance case Res 1 is considered. The complete list of all possible resonances in dynamic systems with cylindric symmetry and the corresponding forms of the amplitude equations are given in Yudovich *et al.* (2003).

We investigate analytically and numerically nonresonance case (Res 0) and compare it with one of six possible resonances (Res 3). Last case corresponds to the amplitude system with most additional terms.

2 Formulation of the problem

We consider the flow of viscous incompressible fluid between two infinite co-axial rigid cylinders of radii r_1 and r_2, which rotate with the angular velocities Ω_1 and Ω_2, respectively. Dimensionless equations of motion (the Navier – Stokes system) can be written in the form

$$\begin{aligned} &\frac{\partial \mathbf{v}}{\partial t} + A\mathbf{v} = -\nabla p - R_1 L(\mathbf{v}, \mathbf{v}), \\ &\nabla \cdot \mathbf{v} = 0, \end{aligned} \tag{1}$$

where $\mathbf{v} = (v_r,\ v_\theta,\ v_z)$ is the velocity of the flow, p is the pressure, $r,\ \theta,\ z$ are cylindric coordinates, the axis z is directed along the axis of the cylinders, $R_1 = \Omega_1 r_1^2 d^2/\nu$ is Reynolds number, ν is kinematic viscosity, $d = \eta - 1$ is dimensionless gap between the cylinders, $\eta = r_2/r_1$ is the ratio of their radii, A – linear, L– nonlinear differential operators. On the solid surfaces non-slip conditions are given.

Let us consider flows with periodic in z velocity and pressure fields, the period $2\pi/\alpha$ being given.

It is well known that for any values of parameters the formulated problem possesses a stationary solution: the Couette flow. In cylindric coordinates ($r,\ \theta,\ z$) its velocity components have the form

$$\mathbf{v}_0(r) = (0,\ ar + b/r,\ 0), a = \frac{R_2 - R_1}{R_1(\eta^2 - 1)}, b = -\frac{R_2 - R_1\eta^2}{R_1(\eta^2 - 1)d^2},$$

and the pressure $p_0 = p_0(r)$ can be determined by a simple integration of the equation $p_0' = -v_{0\theta}^2(r)/r$. Here $R_1 = \Omega_1 r_1^2 d^2/\nu$, $R_2 = \Omega_2 r_2^2 d^2/\nu$ are the Reynolds numbers.

The stability of the Couette flow under the formulated conditions can be studied by the linearization method.

Both the nonlinear (1) and the linearized problems possess a group of symmetry $SO(2) \times O(2)$: they are invariant with respect to rotations L_θ^δ about z-axis, translations L_z^h along the z-axis and inversion J iduced by reflection $z \mapsto -z$, that act over the velocity field as follows

$$\begin{aligned} &(L_\theta^\delta \mathbf{v})(r, \theta, z, t) = \mathbf{v}(r, \theta + \delta, z, t),\ (L_z^h \mathbf{v})(r, \theta, z, t) = \mathbf{v}(r, \theta, z + h, t), \\ &(\mathrm{J}\mathbf{v})(r, \theta, z, t) = (v_r(r, \theta, -z, t), v_\theta(r, \theta, -z, t), -v_z(r, \theta, -z, t)). \end{aligned}$$

So the symmetry of the problem allows us to seek normal oscillations in the form

$$\mathbf{\Phi}(t, r, \theta, z) = e^{i\omega_m t - i(m\theta + k\alpha z)}\varphi(r), \tag{2}$$

where m– azimuthal, k– axial quantum numbers, ω_m – a (real) frequency.

The value of Reynolds number R_1 for which the linearized system admits a non-trivial solution (2) is called the critical Reynolds number $R_{1*}(m, k, \eta,\ R_2, \alpha)$. If the values of quantum numbers m, k and parameters α, η, R_2 are fixed, then there exists an increasing sequence of the critical Reynolds numbers R_{1*}^p

$$R_{1*}^1 < R_{1*}^2 < R_{1*}^3 < \ldots$$

For the fixed values of α, η and (m, k), (n, l) the neutral curves $R_1 = R_{1*}^p(m, k, \eta, R_2, \alpha)$ and $R_1 = R_{1*}^q(n, l, \eta, R_2, \alpha)$ intersect in the point (R_{1*}, R_{2*}) on the plane (R_1, R_2). Here and further indexes p, q are omitted.

3 The systems of amplitude equations and regimes on invariant subspaces

Due to symmetry, the linearized system at each point of intersection (R_{1*}, R_{2*}) (point of codimension - 2 bifurcation) has the following four independent modes

$$\Phi_1(t,r,\theta,z) = e^{i\omega_m t}\Phi_{0m}(r,\theta,z),\ \Phi_2(t,r,\theta,z) = e^{i\omega_m t}\Phi_{1m}(r,\theta,z),$$
$$\Phi_3(t,r,\theta,z) = e^{i\omega_n t}\Phi_{0n}(r,\theta,z),\ \Phi_4(t,r,\theta,z) = e^{i\omega_n t}\Phi_{1n}(r,\theta,z),$$

where $\Phi_{0m}(r,\theta,z) = e^{-i(m\theta+k\alpha z)}\varphi_{0m}(r)$, $\mathbf{\Phi}_{1m}(r,\theta,z) = \mathrm{J}(\Phi_{0m}) = = e^{-i(m\theta-k\alpha z)}\varphi_{1m}(r)$, , $\Phi_{0n}(r,\theta,z) = e^{-i(n\theta+l\alpha z)}\varphi_{0n}(r)$, $\mathbf{\Phi}_{1n}(r,\theta,z) = \mathrm{J}(\Phi_{0n}) = e^{-i(n\theta-l\alpha z)}\varphi_{1n}(r)$.

In a small vicinity $R_1 = R_{1*} + k_1\varepsilon^2$, $R_2 = R_{2*} + k_2\varepsilon^2$ of the point (R_{1*}, R_{2*}) on the plane (R_1, R_2) we search for the asymptotic solution of the nonlinear problem (1) in the form

$$\mathbf{u} = \mathbf{v}_{00} + \varepsilon(\Phi + \Phi^*) + ...$$

Here ε is a small parameter, k_1, k_2 are overcritical constants ($k_1^2 + k_2^2 = 1$), which may be chosen arbitrarily, $\mathbf{v}_{00}$ – the velocity vector of the Couette flow with critical values of the Reynolds numbers, by the sign * we denote complex conjugation, $\Phi = \xi_{0m}(\tau)\Phi_1 + \xi_{1m}(\tau)\Phi_2 + \xi_{0n}(\tau)\Phi_3 + \xi_{1n}(\tau)\Phi_4$, and the unknown complex-valued amplitudes $\xi_{0m}(\tau)$, $\xi_{1m}(\tau)$, $\xi_{0n}(\tau)$, $\xi_{1n}(\tau)$ depend upon slow time $\tau = \varepsilon^2 t$. The terms of order 2 and higher with respect to parameter ε are omitted.

For small ε, with the use of the central manifold theorem, or by averaging in respect to "fast time" t, systems of complex-valued differential equations for the amplitudes $\xi_{0m}, \xi_{1m}, \xi_{0n}, \xi_{1n}$ are construct (Yudovich (1986) – Chossat *et al.* (1994)). The form of the amplitude systems depends upon the ratio between azimuthal m and n, axial k and l quantum numbers, and sometimes also between the phase frequencies ω_m and ω_n of the neutral modes. Besides the generic case, when in the amplitude systems there are presented intrinsic resonances (non-resonant case Res 0), other six cases also exist, which differ by additional resonance summands Yudovich *et al.* (2003).

We consider non-resonant case Res 0 and case of resonance Res 3 ($n = 3m$, $l = k$, $\omega_n = 3\omega_m$).

The amplitude system for Res 3 has the form

$$\begin{aligned}
\xi'_{0m} &= \xi_{0m}(\sigma + A|\xi_{0m}|^2 + B|\xi_{1m}|^2 + C|\xi_{0n}|^2 + D|\xi_{1n}|^2) + Q\xi_{1m}\xi_{0n}\xi^*_{1n} + \\
&\quad + G\xi^*_{0m}\xi^*_{1m}\xi_{0n} + W\xi^{*2}_{1m}\xi_{1n}, \\
\xi'_{1m} &= \xi_{1m}(\sigma + B|\xi_{0m}|^2 + A|\xi_{1m}|^2 + D|\xi_{0n}|^2 + C|\xi_{1n}|^2) + Q\xi_{0m}\xi^*_{0n}\xi_{1n} + \\
&\quad + G\xi^*_{0m}\xi^*_{1m}\xi_{1n} + W\xi^{*2}_{0m}\xi_{0n}, \\
\xi'_{0n} &= \xi_{0n}(\mu + P|\xi_{0m}|^2 + S|\xi_{1m}|^2 + U|\xi_{0n}|^2 + V|\xi_{1n}|^2) + F\xi_{0m}\xi^*_{1m}\xi_{1n} + \widetilde{W}\xi^2_{0m}\xi_{1m}, \\
\xi'_{1n} &= \xi_{1n}(\mu + S|\xi_{0m}|^2 + P|\xi_{1m}|^2 + V|\xi_{0n}|^2 + U|\xi_{1n}|^2) + F\xi^*_{0m}\xi_{1m}\xi_{0n} + \widetilde{W}\xi_{0m}\xi^2_{1m}.
\end{aligned} \tag{3}$$

There are explicit expressions for the coefficients $\sigma, \mu, A, B, C, D, Q, G, W, P, S, U, V, F, \widetilde{W}$ via the eigenfunctions $\Phi_{0m}, \Phi_{1m}, \Phi_{0n}, \Phi_{1n}$ of the linearized problem and the eigenfunctions of its conjugate (see Yudovich *et al.* (2005), Yudovich *et al.* (2006)).

The amplitude system for non-resonant case (Res 0) is obtained from (3) if $Q = G = W = F = \widetilde{W} = 0$.

The system of amplitude equations, as well as the original Navier-Stokes system, possesses the symmetry group $\mathcal{G} = SO(2) \times O(2)$. $\mathcal{G}$-stationary regimes of the system (1) correspond to the $\mathcal{G}$-stationary solutions of the amplitude systems in the invariant subspaces, generated by different one-parameter symmetry subgroups of $\mathcal{G}$. Let us notice, that one parameter symmetry subgroups of $\mathcal{G}$ are isomorphic to the circle group $SO(2)$; the correspondent stationary regimes represent the travelling waves, which can be azimuthal or spiral (travelling along a skew direction in the plane (θ, z)).

Let us list $\mathcal{G}$- stationary solutions of the amplitude system in the non-resonant case Res 0 (see Yudovich *et al*. (2005)):

1. *Inversely connected couple of the m- and n- spiral waves*. Such regimes are situated in one-dimensional invariant subspaces $Y^1(\xi_{jp})$ with one nonzero amplitude ξ_{jp} ($j = 0, 1$; $p = m, n$). They represent secondary periodic solutions of the Navier-Stokes equations (1).

2. *M- and n- azimuthal waves*. These regimes are situated in two-dimensional invariant subspaces $Y^2(\xi_{0p}, \xi_{1p})$, where only two nonzero amplitudes ξ_{0p} and ξ_{1p} ($p = m, n$) exist. J-symmetric periodic regime of the system (1) corresponds to the azimuthal waves. It represents a nonlinear mixture of two J-connected m-spiral waves (or n-waves) travelling along the axis z of cylinders towards each other.

3. *Two two-parameter families of inversely connected double spiral waves*. The unsteady quasi-periodic (with two frequencies) solutions of the system (1) correspond to the double spiral waves. The double spiral waves of the first family (in $Y^2(\xi_{0m}, \xi_{1n})$ or $Y^2(\xi_{1m}, \xi_{0n})$) represent a nonlinear mixture of the m- and n- spiral waves traveling along the axis z towards each other. The solution of the second family (in $Y^2(\xi_{0m}, \xi_{0n})$ or $Y^2(\xi_{1m}, \xi_{1n})$) represent the mixture of spiral waves travelling in the same direction.

4. *Superposition of m- and n- azimuthal waves*. This solution of the amplitude system is in four-dimensional invariant subspace $Y^4(\xi_{0m}, \xi_{1m}, \xi_{0n}, \xi_{1n})$, where $|\xi_{0m}| = |\xi_{1m}|, |\xi_{0n}| = |\xi_{1n}|$. It corresponds to unsteady quasi-periodic regime with two frequencies; it represents a nonlinear mixture of m- and n- azimuthal waves, travelling along the z axis .

5. *Equilibria with one zero amplitude in the invariant three-dimensional space*. Equilibria in the invariant three-dimensional space $Y^3(\xi_{j_1p_1}, \xi_{j_2p_2}, \xi_{j_3p_3})$, where only one amplitude $\xi_{j_4p_4}$ is equal zero ($j_s = 0, 1$; $p_s = m, n$, s=1,2,3,4) correspond to periodic or quasi-periodic (with three frequencies) solutions of the Navier-Stokes system (1).

In addition generic equilibria of the amplitude systems can exist, which do not belong to invariant subspaces.

6. *Generic equilibria*. Generic equilibria correspond to unsteady periodic or quasi-periodic solutions (with four frequencies) of the Navier-Stokes system. The existence and stability of such equilibria is to be investigated numerically.

Existence conditions of listed states depend on coefficients of amplitude equations. These conditions for the first four solution types and eigenvalues of their stability spectrum can be explicitly expressed via these coefficients. Existence and stability of equilibria in $Y^3(\xi_{j_1p_1}, \xi_{j_2p_2}, \xi_{j_3p_3})$ we can checked only numerically.

In the case of resonances Res 3 (see Yudovich *et al*. (2006)) for n- spiral waves the same results are obtained, as for Res 0, except for m- spiral waves, for which only four stability spectrum eigenvalues change. M-azimuth waves on $Y^2(\xi_{0m}, \xi_{1m})$, and a family of J-connected double spiral waves on $Y^2(\xi_{j,m}, \xi_{jn}$ $(j = 0, 1))$ cease to exist. On other two-dimensional invariant subspaces everything is the same as for Res 0, with only four stability spectrum eigenvalues changed.

On three-dimensional invariant subspaces at resonance Res 3 $\mathcal{G}$- stationary solutions of amplitude systems can exist, if phase invariant β (linear combination of phases of complex amplitudes) is equal zero. If $\beta = 0$, then on $Y^4(\xi_{0m}, \xi_{1m}, \xi_{0n}, \xi_{1n})$ superposition of m- and n- azimuthal waves can exist, and, if additional coefficient conditions are also satisfied, other states with all non-zero amplitudes are possible.

Calculations show, that for different sets of the coefficients in the amplitude systems one can meet various cases of existence or non-existence of all listed types of motions. They can be stable or unstable.

4 Conclusions

More complex solutions of the amplitude systems can exist along with the listed $\mathcal{G}$-stationary solutions. Probably, the main question to be solved for the amplitude system is to understand the degree of complexity of its solutions. In particular, one should understand if the chaotic solutions do exist for some values of the parameters. It is also interesting to describe possible bifurcations in the amplitude systems.

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References

Yudovich, V. I. 1986 Transitions and chaos rise in the fluid flows. *6-th Soviet Union Congress on theoretical and applied mechanics. Tashkent, 24–30 September, Abstracts Volume.* 661 (in Russian).

Chossat, P., Demay, Y., Iooss, G. 1987 Interaction de odesazimutaux dans le probleme de Couette - Taylor. *Arch. Rational Mech. Anal.* **99**, 213-248.

Chossat, P., Iooss, G. 1994 The Couette - Taylor problem. *Springer – Verlag, New York.*

Yudovich, V. I., Ovchinnikova, S. N. 2003 Resonances in the intersections of bifurcation in the Couette - Taylor problem. *In Patterns and Waves, A. Abramian, S. Vakulenko, V. Volpert (Eds.), Saint Petersburg*, 55-77.

Yudovich, V. I., Ovchinnikova, S. N., 2005 The intersections of bifurcation in the Couette - Taylor problem. I. Non-resonant case. *VINITI. No. 458-B2005*, 1-33(in Russian).

Yudovich, V. I., Ovchinnikova, S. N., 2006 The intersections of bifurcation in the Couette - Taylor problem. III. Resonances Res 3, Res 4. *VINITI. No. 962-T2006*, 1-33(in Russian).